

Inelastic collapse of a randomly forced particle

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We consider a randomly forced particle moving in a finite region, which rebounds inelastically with coefficient of restitution r on collision with the boundaries. We show that there is a transition at a critical value of r , $r_c \equiv e^{-\pi/\sqrt{3}}$, above which the dynamics is ergodic but beneath which the particle undergoes inelastic collapse, coming to rest after an infinite number of collisions in a finite time. The value of r_c is argued to be independent of the size of the region or the presence of a viscous damping term in the equation of motion.

Randomly accelerated particles have been used as models in many contexts, ranging from the classic problem of Brownian motion [1] to ecological phenomena such as swarming [2]. Theoretical studies have centered on first-passage problems both with [3] and without [4,5] viscous damping or in the presence of a potential [6]. Recently, systems of such particles that collide inelastically have been studied in connection with granular media [7]. These systems are characterized by clustering [8], which is thought to be a collective effect related to inelastic collapse of ballistically moving particles [9].

In this Letter we show that inelastic collapse can occur in a system consisting of a *single* particle. Specifically, we study a randomly forced inelastic particle in a finite box. If the coefficient of restitution governing the collisions with the boundaries is less than some critical value, r_c , the particle will collide with one wall an infinite number of times in a finite time and come to rest at the wall. The value of r_c is shown to be independent of the system size or the presence of viscous damping. This behaviour represents a novel example of a broken ergodicity transition in a single particle, one-dimensional stochastic system. It also has repercussions on the use of randomly-forced inelastic particles as models of granular media.

Consider a particle moving with position x at time t within a region of size l and subject to a random force. The equation of motion is

$$\frac{d^2x}{dt^2} = \eta(t), \quad (1)$$

for $0 < x < l$, where $\eta(t)$ is a Gaussian white noise with correlator $\langle \eta(t_1)\eta(t_2) \rangle = 2\delta(t_1 - t_2)$ and initially we ignore viscous damping. When the particle collides with one of the walls it rebounds inelastically with coefficient of restitution r , i.e., $v_f = -rv_i$, where v_i and v_f are respectively the velocities just before and just after the collision. The random force tends to increase the particle's energy, while the collisions dissipate energy. One might naively expect that the motion would settle into a steady state with a well-defined average energy, except in the pathological cases $r = 0$ (where the particle simply adheres to the first wall it hits) and $r = 1$ (where there is

no dissipation and the particle's energy increases without limit).

Solutions to the Focker-Planck equation for a randomly accelerated particle are technically formidable, even in time-independent situations where the boundary conditions are known [5]. In the present problem, the boundary conditions must be determined self-consistently, so that the flux of particles leaving the system with velocity $-|v|$ is balanced by a flux of particles re-entering the system with velocity $r|v|$. We have not been able to obtain explicit solutions either for the time-dependent case with specific initial conditions, nor for a putative steady state. However, exact arguments about the behaviour of the system may be formulated by mapping the motion onto that of an elastic particle.

We discuss first the semi-infinite case $l = \infty$. The following transformation

$$x \rightarrow x' = r^{-3}x; \quad t \rightarrow t' = r^{-2}t \quad (2)$$

leaves the equation of motion (1) and the variance of the noise invariant. Moreover, Eqs (2) imply $v \rightarrow v' = r^{-1}v$, so if we perform such a rescaling of variables after each collision then in terms of the new variables the motion is that of a randomly accelerated particle which collides elastically with the wall. We define

$$\bar{x} = r^{-3n(\bar{t})}x; \quad d\bar{t} = r^{-2n(\bar{t})}dt, \quad (3)$$

where n is the number of collisions. Furthermore, we can remove the wall and consider the motion of a free particle, in which case n is the number of times the particle has crossed the line $\bar{x} = 0$ and we should take the absolute value of x when transforming back to the original variables. The bar denotes coordinates referring to the fictitious free particle. We can invert Eqn (3) to give:

$$t = \int_0^{\bar{t}} ds r^{2n(s)}. \quad (4)$$

We will see that inelastic collapse occurs when n increases with \bar{t} in such a way that t approaches a finite limit.

To facilitate the discussion, we make a further transformation onto a stationary Gaussian process (SGP) via the following change of variables [10]:

$$X = \left(\frac{3}{2}\right)^{1/2} \frac{\bar{x}}{\bar{t}^{3/2}}; \quad T = \ln \bar{t}. \quad (5)$$

The equation of motion may be written in the form

$$\frac{d^2 X}{dT^2} + 2 \frac{dX}{dT} + \frac{3}{4} X = H(T), \quad (6)$$

where $H(T)$ is a Gaussian white noise with correlator $\langle H(T_1)H(T_2) \rangle = 3\delta(T_1 - T_2)$. The correlation function for $X(T)$ is $\langle X(T_0)X(T_0 + T) \rangle = (3/2)e^{-T/2} - (1/2)e^{-3T/2}$. The average number of returns to the origin in an interval dT of a SGP with correlation function $C(T) \sim 1 - AT^2$ at small T is $d\langle n \rangle = \rho dT$, where $\rho = \sqrt{2A}/\pi$ [10]. Thus $d\langle n \rangle = \rho d\bar{t}/\bar{t}$, so $\langle n \rangle = \rho \ln \bar{t}$ for large \bar{t} . In a naive mean-field approach, we would replace $n(\bar{t})$ by $\langle n(\bar{t}) \rangle$ in Eq. (4) giving $t \approx \int^{\bar{t}} ds s^{\frac{\sqrt{3}}{\pi} \ln r}$. Therefore, when $r < r_c \equiv e^{-\pi/\sqrt{3}}$, $\lim_{\bar{t} \rightarrow \infty} t$ is finite. This means that after a finite time, the particle has collided with the wall an infinite number of times and, since $\langle x^2 \rangle = \langle r^{6n} \bar{x}^2 \rangle$, where $\langle \bar{x}^2 \rangle \sim \bar{t}^3$, the trajectory collapses and the particle adheres to the boundary.

We now turn to an exact calculation of the return velocity distribution after many collisions with the boundary, from which further details of the collapse transition can be extracted. In particular, we will show that the value of r_c predicted by the mean field analysis is correct and that the collapse takes place in a finite time due the presence of a characteristic velocity scale which decays exponentially with the number of bounces. The calculation involves two steps. Firstly, we will determine the speed distribution on the first return to the origin for a particle governed by eqn(1) and released from $x = 0$ with velocity v_0 . Secondly, we will use this distribution as a Green's function to relate the velocity distribution at the boundary after n collisions to that after $n - 1$ collisions. By iterating the resulting recursion relation, the full return distribution after n bounces can be generated.

The velocity distribution on the first return to the boundary can be calculated from the steady state properties of the following escape problem. Particles are injected into the region $x \geq 0$ from the origin at a constant rate and with initial velocity v_0 . They experience a random force, eqn(1), and can only exit from the region at $x = 0$. The steady state flux of particles leaving with speed v is proportional to the first return probability, $P(v|v_0)$, for particles released from the origin with initial velocity v_0 . Calculations of this type have been carried out in the context of Kramers' equation [11] and are known as albedo problems in boundary layer theory. Here we will show that the albedo solution for the undamped, random acceleration model with delta-function injection has a simple analytic form which, to our knowledge, has not been noted previously.

In the steady state, the probability density function $P(x, v)$ corresponding to the Langevin equation (1) obeys the Focker-Planck equation

$$\frac{\partial^2 P(v, x)}{\partial v^2} = v \frac{\partial P(v, x)}{\partial x}. \quad (7)$$

For the escape problem outlined above, this equation must be solved subject to the boundary conditions

$$P(x, v) \rightarrow 0, \quad v \rightarrow \pm\infty \quad (8)$$

$$P(x, v) \rightarrow 0, \quad x \rightarrow +\infty \quad (9)$$

$$P(0, v) = \delta(v - v_0), \quad \text{for } v > 0 \quad (10)$$

with $P(0, v)$ for $v \leq 0$ the unknown function we wish to determine. Using separation of variables, the most general form of the solution to eqn(7) which is continuous and differentiable along the line $v = 0$ and is consistent with the boundary conditions eqn(8) is

$$P(x, v) = \int_{-\infty}^{\infty} d\lambda e^{-\lambda x} a(\lambda) \text{Ai}(-\lambda^{\frac{1}{3}} v), \quad (11)$$

where Ai is the Airy function and $a(\lambda)$ is as yet unknown. Using the orthogonality properties of the Airy functions over the the interval $[-\infty, \infty]$, this equation can be inverted to express $a(\lambda)$ as an integral over $P(x, v)$. Along the line $x = 0$ this reduces to

$$\lambda^{\frac{1}{3}} a(\lambda) = \text{Ai}(-\lambda^{\frac{1}{3}} v_0) + \int_{-\infty}^0 dv v \text{Ai}(-\lambda^{\frac{1}{3}} v) P(0, v), \quad (12)$$

where we have used the boundary condition eqn(10). The requirement that $P(x, v) \rightarrow 0$ for $x \rightarrow \infty$ can only be satisfied if $a(\lambda) = 0$ for all $\lambda < 0$. Imposing this condition on (12) results in an integral equation for the unknown part of the boundary velocity distribution,

$$\text{Ai}(-\lambda^{\frac{1}{3}} v_0) = \int_0^{\infty} dv v \text{Ai}(\lambda^{\frac{1}{3}} v) P(0, -v), \quad (13)$$

which must hold for all $\lambda < 0$.

In the semi-infinite system the only characteristic velocity is v_0 , so this necessarily sets the scale of $P(0, -v)$. Using the relation $P(v|v_0) = -vP(0, -v)$ and the definition $P(v|v_0) = \frac{1}{v_0} f(\frac{v}{v_0})$, we proceed by making the ansatz

$$f(x) = \frac{3}{2\pi} \frac{x^{\frac{3}{2}}}{1 + x^3}, \quad (14)$$

which was motivated by a numerical simulation of the above escape problem. By expressing the Airy function in terms of Bessel functions and performing the integral in (13), one can verify that $P(0, -v)$ obtained from the above ansatz is indeed a solution to (13) for all $\lambda < 0$ as required.

Next we wish to determine how the return velocity distribution evolves as the particle collides many times and is reflected with a coefficient of restitution r at each bounce. Let $P_n(v)$ be the probability density of returning

to the wall with speed v after n collisions. This distribution obeys the recursion equation

$$P_{n+1}(v) = \int_0^\infty \frac{1}{rv'} f\left(\frac{v}{rv'}\right) P_n(v') dv', \quad (15)$$

with $f(x)$ given by (14), as particles incident with speed v are reflected with speed rv . Eqn(15) may be solved by first changing variables to $u = \ln v$ which turns the integral into a convolution. Using standard Fourier techniques, eqn(15), together with the initial condition $P_0(v) = \delta(v - v_0)$, can be solved up to quadrature giving

$$P_n(v) = \frac{1}{(2r)^n v_0} \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{e^{ik(\ln(v/v_0) - n \ln r)}}{[\cosh \frac{\pi k}{3}]^n}. \quad (16)$$

We are now in a position to discuss the collapse transition in some detail. Firstly, one can explicitly calculate the moments of the velocity distribution $P_n(v)$. One finds

$$\left\langle \left(\frac{v}{v_0}\right)^\alpha \right\rangle = \left[\frac{r^\alpha}{2 \cos \frac{\pi}{3}(1 + \alpha)} \right]^n \quad (17)$$

for $-5/2 \leq \alpha \leq 1/2$, whilst moments with α outside this range do not exist. The fluctuations in v are thus extremely large and, as n increases, different moments of v diverge for different values of r . This is because v depends exponentially on n , so the moments are sensitive to the rare events associated with the extremes of the return velocity distribution. However, if one is interested in typical trajectories, the natural variable to consider is $\ln v$ as this grows only linearly with n [12]. Changing variables in eqn(16) to $u = \ln(v/v_0)$ and defining the normalized probability distribution $Q_n(u) = e^u P_n(e^u)$, one finds that in the large n limit,

$$Q_n(u) \sim n^{-\frac{1}{2}} \exp \left[-\frac{9}{8n\pi^2} \left(u - n \ln r - \frac{\pi}{\sqrt{3}} n \right)^2 \right]. \quad (18)$$

From the peak of the distribution we can identify a critical value of r , $r_c = e^{-\pi/\sqrt{3}}$. Note that this value is the same as that predicted by the mean field analysis. Furthermore, the distribution of $\ln v$ is sharply peaked around a value $n \ln(r/r_c)$ with fluctuations of order \sqrt{n} . There is thus a typical, characteristic velocity which behaves like $(r/r_c)^n$. On scaling grounds, one would expect the time intervals between collisions for $r < r_c$ also to decay exponentially with n since $v \sim t^{1/2}$. Consequently, the total elapsed time after an infinite number of bounces will remain finite and, subsequently, the particle will remain at rest on the boundary.

We shall now discuss the case of a particle in a finite box of size l . The trajectory of a particle starting at $x = 0$ with speed v_0 is the same as for the semi-infinite system up to the instant where $x = l$ for the first time, and may be obtained from a trajectory of the equivalent SGP and the transformations (3), (5). In

terms of the variable $X(T)$, the position of the far wall is $L = l \exp(-\frac{3}{2}T - 3n \ln r)$. The quantity $\ln L$ decreases linearly in T , but increases by $-3 \ln r$ at each return of X to zero. Figure 1 shows the logarithm of the envelope X_{\max} of a typical trajectory, defined as the largest value of $|X|$ during the interval between the previous and the next zero, obtained by simulating equation (6), together with the corresponding values of $\ln L(T)$ for three values of r . Because the correlations in the SGP decay exponentially in T , the intervals between zeros will have short-range correlations. It follows that the motion of $\ln L$ is that of a biased random walker, biased towards the origin for $r > r_c$ and away from the origin for $r < r_c$. Meanwhile, the distribution of $|X|$ is very sharply cut off at $|X| \sim 1$. It is known that a random walker with a bias away from the origin has a non-zero probability of never returning to the origin [13]. Thus, for $r < r_c$, the particle has a non-zero probability of never reaching the far wall. If the particle does reach the far wall, it will then have a non-zero probability of never reaching the near wall. The particle will therefore collapse to one of the walls, and for each trajectory $X(T)$ the time taken for the particle to come to rest will be finite. For $r > r_c$ the particle will always reach the other wall in a finite time and the process will repeat indefinitely.

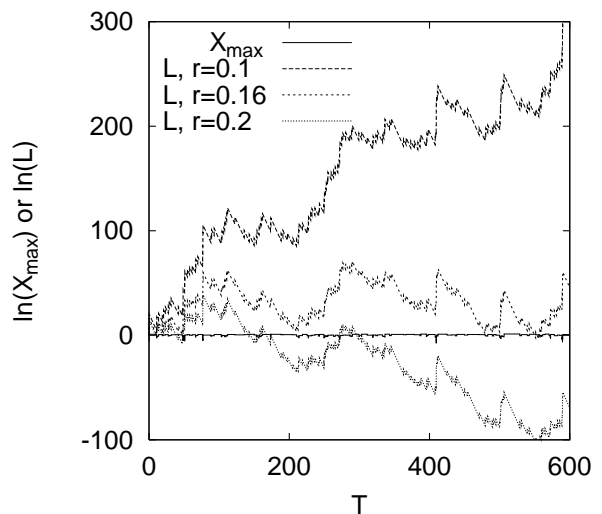


FIG. 1. A semilogarithmic plot of the envelope X_{\max} (see text) of a typical trajectory of the SGP (Eqn. (6)), together with the effective position of the far wall $L(T)$. When $r < r_c = e^{-\pi/\sqrt{3}} \approx 0.163\dots$, the particle has a non-zero chance of never reaching the wall.

There is therefore a symmetry-breaking transition between a steady state where the particle bounces an infinite number of times from either wall, and the collapsed state where the particle bounces a finite number of times from one wall and an infinite number of times from the other in a finite time. Figure 2 shows typical trajectories for four values of r , obtained from simulations of Eqn (6)

and transforming back to the real variables x and t , showing the collapse beneath the critical value $r_c \approx 0.163$.

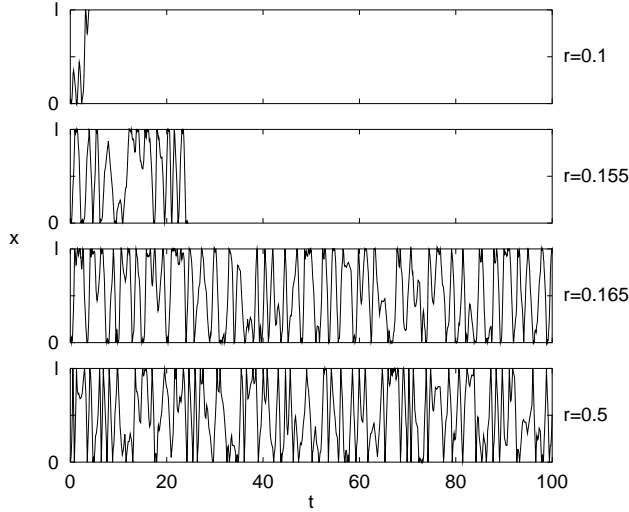


FIG. 2. Typical trajectories of the particle in a finite box for four values of r , showing inelastic collapse for $r < r_c$.

Finally we argue that a collapse transition can occur for particles obeying Kramers' equation of motion,

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} = \eta(t). \quad (19)$$

This equation is often solved perturbatively by expanding about the high viscosity limit [14]. However, if collisions with a boundary are to be included, the effect of the inertial term needs to be treated in a more complete way. We will again consider a particle colliding with a single boundary with coefficient of restitution r . Scaling arguments analogous to those presented above suggest that a collapse transition will take place for all γ at the same value of r as in the undamped case.

We first perform a rescaling of the variables as in (2). The resulting equation of motion has the same form as (19) but with a rescaled γ given by $\gamma' = r^2\gamma$. After many collisions there is an effective time dependent $\bar{\gamma}(\bar{t}) = r^{2n(\bar{t})}\gamma$. From Eqn (4), we see that in the undamped problem for $r < r_c$, $r^{2n(\bar{t})} \rightarrow 0$ faster than $1/\bar{t}$. If one introduces a time dependent γ in eqn(19), a simple scaling analysis shows that it will be irrelevant asymptotically if it too decays faster than $1/t$. Thus, the dissipative system will always be in the collapsed state for $r < r_c$. Now let us assume that the dissipative system can collapse for some $r > r_c$. As the collapsed state is approached, the velocity of the particle goes to zero while the acceleration remains of the order of the noise strength. Consequently the dissipative term γv will become negligible and, as we have assumed $r > r_c$, the particle will ultimately move away from the wall. We thus conclude that the collapse transition will occur when $r = r_c = e^{-\pi/\sqrt{3}}$ for any value of γ [15].

The arguments about collapse of a single particle with a wall can trivially be extended to two inelastic particles, in which case the collapsed state consists of the two particles moving together. Collapse can also be expected in a many-body system, when pairs of particles, and then larger clusters, aggregate. Real systems may deviate from the ideal case studied in this Letter by having short-range correlations in the driving force, or by the coefficient of restitution approaching unity in the small velocity limit. In these cases, one would still expect some remnant of the collapse transition, and this should provide a method for extracting information about the time- and velocity-scales on which such deviations occur. We will discuss these and other related points at greater length elsewhere [16].

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- [1] P. Langevin, *Comptes Rendues* **146**, 530 (1908).
- [2] D. Grunbaum, *J. Theor. Biol.* **33**, 139 (1995).
- [3] T. W. Marshall and E. J. Watson, *J. Phys. A* **18**, 3531 (1985).
- [4] Y. Sinai, *Theor. Math. Phys.* **90**, 323 (1992), T. W. Burkhardt, *J. Phys. A* **26**, L1157 (1993).
- [5] J. Masoliver and J. M. Porrà, *Phys. Rev. Lett.* **75**, 189 (1995).
- [6] P. W. Duck, T. W. Marshall and E. J. Watson, *J. Phys. A* **19**, 3545 (1986).
- [7] For a review see H. M. Jaeger, S. R. Nagel and R. P. Behringer, *Rev. Mod. Phys.* **68**, 1259 (1996); See also D. J. Hornbaker, R. Albert, A.-L. Barabasi and P. Schiffer, *Nature (London)* **387**, 765 (1997).
- [8] D. R. M. Williams and F. C. MacKintosh, *Phys. Rev. E* **54**, R9 (1996); M. R. Swift, M. Boamfà, S. J. Cornell and A. Maritan, unpublished.
- [9] S. McNamara and W. R. Young, *Phys. Fluids A* **4**, 496 (1992); *Phys. Rev. E* **50**, R28 (1994).
- [10] S. N. Majumdar, C. Sire, A. J. Bray, and S. J. Cornell, *Phys. Rev. Lett.* **77**, 2867 (1996); B. Derrida, V. Hakim, and R. Zeitak, *ibid.* 2871.
- [11] A. J. Kainz and U. M. Titulaer, *J. Phys. A* **24**, 4677 (1991) and refs. therein.
- [12] This is analogous to considering the free energy, not the partition function, in disordered equilibrium systems.
- [13] W. Feller, *An introduction to probability theory and its applications*, Vol. 1 (Wiley, N. Y., 1968).
- [14] H. Risken, *The Fokker-Planck Equation: Method of Solution and Applications* (Springer-Verlag, 1989).
- [15] The same scaling approach shows that an attractive linear potential at the wall will also not change the value of r_c .
- [16] S. J. Cornell, M. R. Swift and A. J. Bray, unpublished.